First-Order Corrections to Approximate Solutions to Two-Point Boundary-Condition Problems

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A method, applicable to real time guidance, is developed for accurate solution to exo-atmospheric space flight optimization problems. In principle the method is applicable to many other two-point boundary-condition (TPBC) problems. The first step of the method is the iterative solution (using a shooting method) of a TPBC problem with differential equations simplified so that they may be solved analytically by means of a single closed-form solution over each stage of the flight. The second step is the addition of a closed-form correction to the solution to the TPBC problem obtained in the first step. The correction accounts (to first-order accuracy) for the errors due to the aforementioned simplifications. Numerical results are given for several orbital injection problems; for example, the costate was obtained accurately to almost three significant figures for a maneuver requiring a flight plane inclination change of 5° and an altitude change of over 90 km. In another problem the first-order correction reduced the error in the final time from 13.5 to 0.3 sec. Partial derivatives, required for the solution of the TPBC problem, are computed in closed form.

Introduction

A METHOD will be developed for obtaining a first-order correction to an approximate solution to a TPBC problem. The method is applied to the problem of optimizing, with respect to payload, the vacuum flight of a rocket. In order for the procedure to be applicable, it must be possible to replace the differential equations with approximate differential equations which can be integrated in closed form. The method calls for the introduction of a parameter ε into the original differential equations in such a manner that, when ε is set equal to zero, the approximate differential equations result and, when ε is set equal to unity, the original differential equations are obtained. The differential equations, with ε introduced in this manner, may be written in the form $\dot{x} = f(x, t, \varepsilon)$ where x and f are column vectors of n components.

First, the complete solution to the TPBC problem, using the approximate differential equations $\dot{x} = f(x, t, o)$, is obtained by iterative means. Afterwards a first-order correction is added to the solution. Consider the vector x_0 of initial values to be a function $x(\varepsilon)$, where $x_0(\varepsilon)$ corresponds to the converged solution to the TPBC problem associated with the equations $\dot{x} = f(x, t, \varepsilon)$. Also consider final time t_F to be a function of ε. Taylor series expansions about $\varepsilon = 0$, evaluated at $\varepsilon = 1$ and truncated after first-order terms, are employed:

$$x_0(1) \cong x_0(0) + 1 \cdot \left.\frac{dx_0}{d\varepsilon}\right|_{\varepsilon=0}, \quad t_F(1) \cong t_F(0) + 1 \cdot \left.\frac{dt_F}{d\varepsilon}\right|_{\varepsilon=0} \tag{1}$$

General Method for Obtaining a First-Order Correction

Consider the equations $\dot{x} = f(x, t, \varepsilon)$ with initial-point boundary conditions $p(x_0) = 0$, $t_0 = 0$ and final-point boundary-conditions $g(x_F, t_F) = 0$. It is assumed that there are a total of $n+2$ independent boundary conditions. Assume that a solution $\{x_0(\varepsilon), t_F(\varepsilon)\}$ to the TPBC problem exists for any value of ε in the closed interval $[0, 1]$. The subscripts 0 and F denote initial and final values, respectively.

Define $x(x_0, t, \varepsilon)$ to be the solution to the equations $\dot{x} = f$ with initial values x_0 and parameter ε. Define a new function $x_F^*(\varepsilon) \triangleq x[x_0(\varepsilon), t_F(\varepsilon), \varepsilon]$. Then the boundary conditions are identities in $\varepsilon: p[x_0(\varepsilon)] \equiv 0$ and $g[x_F^*(\varepsilon), t_F(\varepsilon)] \equiv 0$. The total derivatives of p and g with respect to ε must also be identically zero:

$$\frac{\partial p}{\partial x_0} x_{0\varepsilon} \equiv 0, \qquad \frac{\partial g}{\partial t_F} t_{F\varepsilon} + \frac{\partial g}{\partial x_F} x_{F\varepsilon}^* \equiv 0 \tag{2}$$

where

$$x_{F\varepsilon}^* = \left.\left(\frac{\partial x}{\partial x_0} x_{0\varepsilon} + \frac{\partial x}{\partial t_F} t_{F\varepsilon} + \frac{\partial x}{\partial \varepsilon}\right)\right|_{t=t_F} \tag{3}$$

and $\partial x/\partial t_F|_{t=t_F} = \dot{x}_F$. For brevity the symbol $x_{F\varepsilon}^*$ is being used, for example, to signify $dx_F^*/d\varepsilon$.

Substitution of the right-hand member of Eq. (3) into (2) yields

$$\frac{\partial p}{\partial x_0} x_{0\varepsilon} \equiv 0, \quad \frac{\partial g}{\partial x_F}\frac{\partial x_F}{\partial x_0} x_{0\varepsilon} + \left(\frac{\partial g}{\partial x_F}\dot{x}_F + \frac{\partial g}{\partial t_F}\right) t_{F\varepsilon} = -\left.\frac{\partial g}{\partial x_F}\frac{\partial x}{\partial \varepsilon}\right|_{t=t_F} \tag{4}$$

If all coefficients are evaluated at $\varepsilon = 0$, Eqs. (4) may be solved simultaneously for $x_{0\varepsilon}(0)$ and $t_{F\varepsilon}(0)$, which are employed in Eqs. (1).

A closed form expression for the vector $\partial x/\partial \varepsilon$ in Eqs. (4) may often be determined by integrating in closed-form the equations of variation

$$\dot{x}_\varepsilon = (\partial f/\partial x)x_\varepsilon + f_\varepsilon \quad [x_\varepsilon(t_0) = 0] \tag{5}$$

Similarly, a closed-form expression for the matrix $\partial x_F/\partial x_0$ may be obtained by differentiating the closed-form expression for x. It may also often be determined by integrating the equations of variation

$$\frac{d}{dt}\frac{\partial x}{\partial x_0} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial x_0} \qquad \left[\left.\frac{\partial x}{\partial x_0}\right|_{t_0} = I\right] \tag{6}$$

The latter approach is taken for the problem to be treated in the following section, because the solution to Eqs. (6) will be a special case of the solution to Eq. (5).

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Solution for Vacuum Flight

A special case will now be considered; namely, the problem of optimal flight of a rocket with constant thrust magnitude F through a vacuum. The equations defining the optimal flight may be written as follows[1]:

$$\dot{\bar{V}} = (F/m)\hat{\lambda} + \bar{G}(\bar{R}), \quad \dot{\bar{R}} = \bar{V}, \quad \dot{m} = -\beta$$
$$\dot{\bar{\lambda}} = -\bar{p}, \qquad \dot{\bar{p}} = -[(\partial/\partial\bar{R})\bar{G}(\bar{R})]^T\bar{\lambda} \tag{7}$$

where $\bar{R}$ and $\bar{V}$ are the position and velocity of the vehicle, m and β are the mass and constant fuel burning rate magnitude, $\bar{\lambda}$ and $\bar{p}$ are the costate vectors, and $\bar{G}$ is gravitational acceleration.

The equations

$$\ddot{\bar{R}} = (F/m)\hat{\lambda} + \bar{G}(t) + \varepsilon[\bar{G}(\bar{R}) - \bar{G}(t)]$$
$$\ddot{\bar{\lambda}} = \varepsilon[(\partial/\partial\bar{R})\bar{G}(\bar{R})]^T\bar{\lambda}, \qquad \dot{m} = -\beta \tag{8}$$

reduce to Eq. (7) when $\varepsilon = 1$. The symbol $\bar{G}(t)$ signifies a polynomial in time which approximates $\bar{G}(\bar{R})$. It may be obtained by expanding $\bar{G}(\bar{R})$ about the initial point in a Taylor series in time truncated after several terms.

When $\varepsilon = 0$, Eqs. (8) reduce to

$$\ddot{\bar{R}} = (F/m)\hat{\lambda} + \bar{G}(t), \quad \ddot{\bar{\lambda}} = 0 \tag{9}$$

which can be integrated in closed form.[2-4] Since $\bar{G}$ does not vary strongly with respect to $\bar{R}$ and is not directly dependent upon m and $\bar{\lambda}$, a solution to Eqs. (9) is often a good approximation to that of Eqs. (8).

The first of Eqs. (9) may be written as

$$\ddot{\bar{R}} = \frac{F(\bar{\lambda}_0 + t\dot{\bar{\lambda}})}{(m_0 + \dot{m}t)|\bar{\lambda}|} + \bar{G}(t) \tag{10}$$

where $|\bar{\lambda}| = (at^2 + bt + c)^{1/2}$, $a = |\dot{\bar{\lambda}}|^2$, $b = 2\bar{\lambda}_0 \cdot \dot{\bar{\lambda}}$, $c = |\bar{\lambda}_0|^2$.

It has been shown[2-4] that the solution to Eqs. (10) is

$$\dot{\bar{R}} = \dot{\bar{R}}_0 + \frac{F}{\dot{m}}[(\ln q_2)\hat{P} - (\ln q_1)\hat{\lambda}] + \int_0^t \bar{G}\,dt$$
$$\bar{R} = \bar{R}_0 + t\dot{\bar{R}}_0 + \int_0^t\int_0^\tau \bar{G}\,dt\,d\tau \tag{11}$$
$$+ \frac{F}{\dot{m}}\left\{\frac{m}{\dot{m}}(\ln q_2)\hat{P} + \frac{\ln q_1}{|\dot{\bar{\lambda}}|}\left[\frac{\bar{P}}{\dot{m}} - (\hat{\lambda}\cdot\bar{\lambda})\hat{\lambda}\right] + \frac{|\bar{\lambda}_0| - |\bar{\lambda}|}{|\dot{\bar{\lambda}}|}\hat{\lambda}\right\}$$

where $\bar{P} = \dot{m}\bar{\lambda}_0 - m_0\dot{\bar{\lambda}}$, $q_1 = (\bar{\lambda}\cdot\dot{\bar{\lambda}} - |\dot{\bar{\lambda}}||\bar{\lambda}|)/(\bar{\lambda}_0\cdot\dot{\bar{\lambda}} - |\dot{\bar{\lambda}}||\bar{\lambda}_0|)$, and $q_2 = (m_0/m)(\bar{\lambda}\cdot\bar{P} - |\bar{\lambda}||\bar{P}|)/(\bar{\lambda}_0\cdot\bar{P} - |\bar{\lambda}_0||\bar{P}|)$. The first integral of Eqs. (10) is essentially obtained by determining

$$J_1 = \int_{m_0}^m \frac{dm}{|\bar{\lambda}|}, \qquad J_2 = \int_{m_0}^m \frac{dm}{m|\bar{\lambda}|}$$

with $|\bar{\lambda}|$ expressed in terms of m rather than t. It may be shown[3] that $J_1 = -\dot{m}(\ln q_1)/(a)^{1/2}$, and $J_2 = \dot{m}(\ln q_2)/|\bar{P}|$. The second integral is obtained by integrating J_1 and J_2. It may be shown[3] that

$$\int_0^t J_1\,dt = -\dot{m}[(t + \bar{\lambda}_0\cdot\dot{\bar{\lambda}}/|\dot{\bar{\lambda}}|^2)\ln q_1 + (|\bar{\lambda}| - |\bar{\lambda}_0|)/|\dot{\bar{\lambda}}|]/(a)^{1/2} \tag{12}$$

$$\int_0^t J_2\,dt = (\ln q_1)/(a)^{1/2} + m(\ln q_2)/|\bar{P}| \tag{13}$$

The equations of variation (5), corresponding to Eqs. (8) for $\varepsilon = 0$, are

$$\ddot{\bar{R}}_\varepsilon = [F/(m|\bar{\lambda}|)](I - \hat{\lambda}\hat{\lambda}^T)\bar{\lambda}_\varepsilon + \bar{G}(\bar{R}) - \bar{G}(t)$$
$$\ddot{\bar{\lambda}}_\varepsilon = [(\partial/\partial\bar{R})\bar{G}(\bar{R})]^T\bar{\lambda} \tag{14}$$

The latter equations can be integrated in closed form provided $\bar{G}(\bar{R})$ and $\partial\bar{G}/\partial\bar{R}$ are approximated as polynomial functions of time which are obtained from the converged solution to the TPBC problem associated with the approximate equations (9). (The latter polynomial representations do not need to be highly accurate, because they are employed in obtaining a small correction to $\bar{\lambda}_0$, $\dot{\bar{\lambda}}_0$, and t_F. The errors in these representations will not be corrected.) Since $\bar{\lambda}_\varepsilon$ can be obtained as a polynomial function of time from the second of Eqs. (14), the integration of Eqs. (14) will be essentially accomplished if the single and double integrals of $A_N = (t^N/m)(I - \hat{\lambda}\hat{\lambda}^T)/|\bar{\lambda}|$ are obtained for various non-negative integers N. It can be easily shown that

$$\int_0^t A_N\,dt = \sum_{i=0}^{2} I_{P,N+i}a_i + \sum_{i=0}^{3} I_{Q,N+i}b_i$$
$$\int_0^t\int_0^\tau A_N\,dt\,d\tau = \sum_{i=0}^{2}\left(\int_0^t I_{P,N+i}\,dt\right)a_i + \sum_{i=0}^{3}\left(\int_0^t I_{Q,N+i}\,dt\right)b_i$$

where

$$I_{PM} = \int_0^t \frac{t^M}{m|\bar{\lambda}|}dt, \qquad I_{QM} = \int_0^t \frac{t^M}{|\bar{\lambda}|^3}dt$$
$$a_0 = F(I - a'\bar{\lambda}_0\bar{\lambda}_0^T) \qquad b_0 = -Fc'\bar{\lambda}_0\bar{\lambda}_0^T$$
$$a_1 = -Fa'B \qquad b_1 = -F(b'\bar{\lambda}_0\bar{\lambda}_0^T + c'B)$$
$$a_2 = -Fa'\dot{\bar{\lambda}}\dot{\bar{\lambda}}^T \qquad b_2 = -F(b'B + c'\dot{\bar{\lambda}}\dot{\bar{\lambda}}^T)$$
$$B = \bar{\lambda}_0\dot{\bar{\lambda}}^T + \dot{\bar{\lambda}}\bar{\lambda}_0^T \qquad b_3 = -Fb'\dot{\bar{\lambda}}\dot{\bar{\lambda}}^T$$
$$a' = \frac{\dot{m}^2}{am_0^2 + c\dot{m}^2 - bm_0\dot{m}}, \quad b' = -\frac{aa'}{\dot{m}}, \quad c' = \frac{1 - ca'}{m_0}$$

Next it will be shown how I_{PM} and O_{QM} and their integrals can be obtained recursively. Use will be made of the integrals

$$I_{RM} = \int_0^t \frac{t^M}{|\bar{\lambda}|}dt, \quad I_{SM} = \int_0^t t^M|\bar{\lambda}|\,dt$$

From the expression for J_1 and integral tables, $I_{RO} = -\ln q_1/(a)^{1/2}$ and

$$I_{R1} = \frac{1}{\dot{m}^2}\int_0^t \frac{m}{|\bar{\lambda}|}dm - \frac{m_0}{\dot{m}^2}\int_0^t \frac{dm}{|\bar{\lambda}|}$$
$$= \frac{|\bar{\lambda}| - |\bar{\lambda}_0|}{\tilde{a}\dot{m}^2} + \frac{1}{\dot{m}(a)^{1/2}}\left(\frac{\tilde{b}}{2\tilde{a}} + m_0\right)\ln q_1$$

where $\tilde{a} = a/\dot{m}^2$ and $\tilde{b} = -2am_0/\dot{m}^2 + b/\dot{m}$. From any table of integrals, for $M = 2, 3, \ldots$,

$$I_{RM} = \frac{t^{M-1}|\bar{\lambda}|}{Ma} - \frac{(2M-1)b}{2Ma}I_{R,M-1} - \frac{(M-1)c}{Ma}I_{R,M-2} \tag{15}$$

From the expression for J_2, $I_{PO} = \ln q_2/|\bar{P}|$ and I_{PM} $(M = 1, 2, \ldots)$ can be broken up as

$$I_{PM} = (I_{R,M-1} - m_0 I_{P,M-1})/\dot{m} \tag{16}$$

From the integral tables

$$I_{QO} = \frac{1}{4ac - b^2}\left(\frac{4at + 2b}{|\bar{\lambda}|} - \frac{2b}{|\bar{\lambda}_0|}\right) \tag{17a}$$

$$I_{Q1} = \frac{-1}{4ac - b^2}\left(\frac{2bt + 4c}{|\bar{\lambda}|} - \frac{4c}{|\bar{\lambda}_0|}\right) \tag{17b}$$

$$I_{QM} = -\frac{t^{M-1}}{a|\bar{\lambda}|} - \frac{b}{2a}I_{Q,M-1} + \frac{M-1}{a}I_{R,M-2} \tag{17c}$$

for $M = 2, 3, \ldots$.

Using integration by parts with $u = |\bar{\lambda}|$, it may be shown that $I_{SM} = (|\bar{\lambda}|t^{M+1} - aI_{R,M+2} - bI_{R,M+1})/(M+1)$ for $M = 0, 1, \ldots$. Moreover, from Eq. (12),

$$\int_0^t I_{RO}\,dt = -\frac{1}{(a)^{1/2}}\left[\left(t + \frac{\bar{\lambda}_0\cdot\dot{\bar{\lambda}}}{|\dot{\bar{\lambda}}|^2}\right)\ln q_1 + \frac{|\bar{\lambda}| - |\bar{\lambda}_0|}{|\dot{\bar{\lambda}}|}\right]$$

$$\int_0^t I_{R1}\,dt = -\frac{|\lambda_0|t}{a} + \frac{1}{a}I_{SO} - \frac{a}{\dot{m}}\left(\frac{\tilde{b}}{2\tilde{a}} + m_0\right)\int_0^t I_{RO}\,dt$$

and from Eq. (15), for $M = 2, 3, \ldots$,

$$\int_0^t I_{RM}\,dt = \frac{1}{Ma}I_{S,M-1} - \frac{(2M-1)b}{2Ma}\int_0^t I_{R,M-1}\,dt - \frac{(M-1)c}{Ma}\int_0^t I_{R,M-2}\,dt$$

is obtained. From Eq. (13),

$$\int_0^t I_{PO}\,dt = [(\ln q_1)/(a)^{1/2} + m(\ln q_2)/|\bar{P}|]/\dot{m}$$

and from Eq. (16) it follows, for $M = 1, 2, \ldots$, that

$$\int_0^t I_{PM}\,dt = \left(\int_0^t I_{R,M-1}\,dt - m_0 \int_0^t I_{P,M-1}\,dt\right)\Big/\dot{m}$$

From Eqs. (17)

$$\int_0^t I_{Q0}\,dt = (4aI_{R1} + 2bI_{RO} - 2bt/|\bar{\lambda}_0|)/(4ac - b^2)$$

$$\int_0^t I_{Q1}\,dt = -(2bI_{R1} + 4cI_{RO} - 4ct/|\bar{\lambda}_0|)/(4ac - b^2)$$

$$\int_0^t I_{QM}\,dt = -\frac{1}{a}I_{R,M-1} - \frac{b}{2a}\int_0^t I_{Q,M-1}\,dt + \frac{M-1}{a}\int_0^t I_{R,M-2}\,dt$$

$(M = 2, 3, \ldots)$.

The equations of variation (6), corresponding to Eqs. (8) for $\varepsilon = 0$, are

$$\frac{d^2}{dt^2}\frac{\partial \bar{R}}{\partial \bar{y}} = \frac{F}{m|\bar{\lambda}|}(I - \hat{\lambda}\hat{\lambda}^T)\frac{\partial \bar{\lambda}}{\partial \bar{y}}, \qquad \frac{d^2}{dt^2}\frac{\partial \bar{\lambda}}{\partial \bar{y}} = 0 \tag{18}$$

where $\bar{y} = (\bar{V}, \bar{R}, \bar{\lambda}, \bar{\rho})$. It can be seen that the integrals of Eqs. (18) may be easily obtained once the integrals of Eqs. (14) have been developed.

Computational Sequence

In summary, one first solves, by iterative means, the TPBC problem associated with Eqs. (9), which have the closed-form solution (11). Next one obtains a closed form solution to the equations of variation (14) and (18). Then the linear equations (4) are solved for $x_{0\varepsilon}$ and $t_{F\varepsilon}$, which are employed in Eqs. (1) in order to obtain corrected values of x_{i0} and t_F.

Staging

Suppose it is necessary to change one or more of the quantities, F, β, and m discontinuously at specified times $t_1, t_2, \ldots, t_{L-1}$. Let $t_L = t_F$. Then one must apply the closed-form solution over each interval (t_i, t_{i+1}) even though (in this analysis) it is assumed in obtaining the closed-form solutions that $\bar{\lambda}$ is linear over the entire flight. At the staging times t_i, the partial derivatives $\partial\bar{R}/\partial\bar{\lambda}_0$, $\partial\bar{R}/\partial\bar{\rho}_0$, $\partial\bar{V}/\partial\bar{\lambda}_0$, and $\partial\bar{V}/\partial\bar{\rho}_0$ are discontinuous, but they may be computed by means of the recursive formulas which will now be developed.

Let $\bar{R}_i$, for example, signify $\bar{R}(t_i)$. When $\varepsilon = 0$, $\bar{\lambda}_{i+1} = \bar{\lambda}_i - (t_{i+1} - t_i)\bar{\rho}$, where $\bar{\rho} \equiv$ constant, $\bar{R}_{i+1} = \bar{R}_i + (t_{i+1} - t_i)\bar{V}_i +$ (function of $\bar{\lambda}_i$ and $\bar{\rho}_i$), and $\bar{V}_{i+1} = \bar{V}_i +$(function of $\bar{\lambda}_i$ and $\bar{\rho}_i$). Therefore,

$$\partial\bar{R}_{i+1}/\partial\bar{\lambda}_0 = \partial\bar{R}_i/\partial\bar{\lambda}_0 + (t_{i+1} - t_i)\,\partial\bar{V}_i/\partial\lambda_0 + \partial\bar{R}_{i+1}/\partial\bar{\lambda}_i$$

$$\partial\bar{R}_{i+1}/\partial\bar{\rho}_0 = \partial\bar{R}_i/\partial\bar{\rho}_0 + (t_{i+1} - t_i)\,\partial\bar{V}_i/\partial\bar{\rho}_0 - (t_i - t_0)\,\partial\bar{R}_{i+1}/\partial\bar{\lambda}_i + \partial\bar{R}_{i+1}/\partial\bar{\rho}_i$$

$$\partial\bar{V}_{i+1}/\partial\bar{\lambda}_0 = \partial\bar{V}_i/\partial\bar{\lambda}_0 + \partial\bar{V}_{i+1}/\partial\bar{\lambda}_i$$

$$\partial\bar{V}_{i+1}/\partial\bar{\rho}_0 = \partial\bar{V}_i/\partial\bar{\rho}_0 - (t_i - t_0)\,\partial\bar{V}_{i+1}/\partial\bar{\lambda}_i + \partial\bar{V}_{i+1}/\partial\bar{\rho}_i$$

Numerical Results

The procedure described has been applied to several exo-atmospheric space flight problems calling for insertion of a space vehicle into elliptical Earth orbits with specified inclinations i, eccentricities, and semimajor axes. The insertion is at perigee, and the optimization is with respect to payload delivered into orbit. The final end conditions are

$$\bar{R}_F \cdot \bar{R}_F = R_P{}^2, \quad \bar{V}_F \cdot \bar{V}_F = V_P{}^2, \quad \bar{R}_F \cdot \bar{V}_F = 0$$

$$\hat{N} \cdot (\bar{R}_F \times \bar{V}_F) = R_P V_P \cos i$$

$$\hat{N} \cdot (\dot{\bar{\lambda}}_F \times \bar{R}_F + \bar{\lambda}_F \times \bar{V}_F) = 0 \tag{19}$$

$$(\bar{R}_F \times \bar{V}_F) \cdot (\dot{\bar{\lambda}}_F \times \bar{R}_F + \bar{\lambda}_F \times \bar{V}_F) = 0 \tag{20}$$

where R_P and V_P are the magnitudes of the radius vector and velocity at perigee, and $\hat{N}$ is a north-pointing vector. Conditions (19) and (20) are transversality conditions determined from the calculus of variations. Initial values were given for $\bar{R}$ and $\bar{V}$, and there was a scaling condition $\bar{\lambda}_0 \cdot \bar{\lambda}_0 = 1$. The gravitational acceleration was expanded in a third-order Taylor series in time about the initial point of each stage of the trajectory. This was done for every iteration. Three cases, described below, are considered. In addition, numerous cases not presented here were tried. In some of the more complex maneuvers the iterations upon the boundary-condition problems failed to converge. However, these failures reflect upon the numerical technique for solving the TPBC problem not upon the method for obtaining a first-order correction to the solution to the TPBC problem.

Case 1

The initial and final conditions are

m_0 = 600,000 kg	$i_0 \cong 55.0°$
F = 200,000 N	Lat$_0$ = latitude = 36.1857°
$\lvert\dot{m}\rvert$ = 50 kg/sec	R_P = 6558.4953 km
$\lvert\bar{R}_O\rvert$ = 6465.8934 km	V_P = 7950.00 m/sec
$\lvert\bar{V}_O\rvert$ = 7879.4175 m/sec	i_F = 50°
γ_0 = flight-path angle = −0.00056°	

The problem calls for a plane change of 5° and an altitude change of 92.6 km. It is a single-stage problem. The numerical results are shown in Table 1. The closed-form solution yields initial values for the costate accurate to one significant figure counting the indicated zeros. The corrected values are accurate to almost three significant figures. The error in the final time was reduced from 3.1 to 0.4 sec.

Table 1 Costate and t_F for case 1

	$\bar{\lambda}_0$	$\bar{\rho}_0$	t_F
True	0.3613975 0.9324115 0.0007410	1.177054×10^{-3} 1.649808×10^{-3} 0.175164×10^{-3}	482.8861 sec
Closed form solution	0.3399935 0.9404206 −0.0036620	1.002866×10^{-3} 1.893104×10^{-3} 0.105612×10^{-3}	485.9867 sec
Corrected	0.3595961 0.9331018 −0.0034045	1.175721×10^{-3} 1.640601×10^{-3} 0.164852×10^{-3}	483.2437 sec

Case 2

The initial and final conditions are the same as in Case 1 except that F = 100,000 N and i_F = 55°. Only a single stage is employed. Because of the low thrust, a rather large central angle of 30° is subtended by the trajectory from the center of the Earth. The numerical results are shown in Table 2. The closed-form solution yields accuracy in the costate approaching two figures. Time t_F is off by 13.5 sec. The corrected solution

Table 2 Costate and t_F for case 2

	$\bar{\lambda}_0$	$\bar{\rho}_0$	t_F
True	0.8929618 0.0286671 0.4492186	0.00432504 0.00013325 0.00203561	435.183 sec
Closed form solution	0.8869390 0.0290621 0.4609714	0.00411382 0.00013171 0.00206135	448.650 sec
Corrected	0.8949155 0.0285374 0.4453220	0.00430998 0.00013124 0.00198957	434.873 sec

has accuracy in the multipliers approaching three figures, and the error in t_F is 0.3 sec.

Case 3

The initial and final conditions are

m_0 = 160,113.69 kg
F (first stage) = 2,500,000 N
F (second stage) = 500,000 N
$|\dot{m}|$ (first stage) = 500 kg/sec
$|\dot{m}|$ (second stage) = 100 kg/sec
Δm (at stage time of 115.27493 sec) = −33,144.51 kg
$|\bar{R}_O|$ = 6385.8032 km i_0 34.6°
$|\bar{V}_O|$ = 709.34548 m/sec $\text{Lat}_0 \cong$ latitude = 28.40°
$\gamma_0 \cong$ flight-path angle = 29.52°
R_P = 6577.0153 km, V_P = 7979.4236 m/sec, i_F = 35.0°

This case represents a two-stage flight of a space shuttle starting shortly after the time of maximum dynamic pressure. Although the aerodynamic forces are still quite significant at the initial time, they have been ignored in this study. The total change in altitude is 191 km, and the change in the flight plane inclination is 0.6°. The numerical results for Case 3 are shown in Table 3. The closed-form solution has an accuracy of one significant figure, counting the indicated zeros. The corrected solution is accurate to two figures.

Table 3 Costate and t_F for case 3

	$\bar{\lambda}$	$\bar{\rho}$	t_F
True	0.7734077 0.3233009 0.5452679	1.940864×10^{-3} 0.030496×10^{-3} 0.037415×10^{-3}	617.407 sec
Closed form solution	0.7216738 0.3526691 0.5956605	1.433160×10^{-3} 0.261997×10^{-3} 0.435889×10^{-3}	617.270 sec
Corrected	0.7714358 0.3245130 0.5473373	1.917814×10^{-3} 0.054779×10^{-3} 0.078948×10^{-3}	617.688 sec

Conclusions

The general theory presented in this paper is applicable to those TPBC problems for which the equations $\dot{x} = f(x, t, \varepsilon)$ can be integrated in closed form with $\varepsilon = 0$ and for which the equations of variation (5) and (6), can be integrated in closed form along the resulting solution $x(t)$. It is also required that the $\varepsilon = 0$ solution to the TPBC problem be at least a rough approximation to the $\varepsilon = 1$ solution (i.e., to the "true" solution).

It is beyond the scope of this paper to give detailed developments, but there are a number of feasible applications and extensions of the general theory. For example, it can be applied if the thrust magnitude F, appearing in Eqs. (8) is approximated as a polynomial function of time. In fact, the symbol F can be a matrix of polynomial functions of time. It can be shown that such a formulation is desirable if one wishes to incorporate aerodynamic effects.[5]

A technique employed by the authors of Ref. 2 can be employed in order to allow for nonlinear $\bar{\lambda}$. One may employ multiple legs and let $\dot{\bar{\lambda}}$ be discontinuous at junction points. Moreover, $\bar{G}(\bar{R})$ may be expanded about the initial point of each leg. It is also feasible to introduce intermediate coast arcs. However, these latter extensions require some additional formulation. A simple extension allows for the parameter ε to occur in the end conditions as well as in the differential equations.

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